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LETTER TO THE EDITOR

Table Look-Up Method for the Evaluation of Functions

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The table look-up method for finding values of functions is probably as old as mathematics itself. Tables of logarithmic and trigonometric functions that were used everywhere 50 or 60 years ago have been replaced by electronic calculators. The table look-up method, however, has survived and it is still used in computer programs when the speed of computations is extremely important.

Suppose that we have to evaluate a continuous function f very fast for a large number of values of the variable x in $[0, 1]$. With a fixed integer N, we first create a table of values

$$f(0),\ f\left(\frac{1}{N}\right),\ ...,\ f\left(\frac{N-1}{N}\right),\ f(1).$$

Then, for an arbitrary x in $[0, 1]$, to get an approximate value for $f(x)$ we have only to compute $[Nx]$, the integral part of Nx, or, using programming language, we have to convert the floating point number Nx into the integer $[Nx]$. If that integer is Q, the approximate value of $f(x)$ is found at position $Q+1$ in the table.

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From the approximation-theoretic point of view, the look-up method can be described as approximation of $f(x)$ by $f([Nx]/N)$. If we assume that the function f satisfies the Lipschitz condition

$$|f(x)-f(y)| \leqslant |x-y|, \tag{1}$$

we already have the error estimate

$$\left| f\left(\frac{[Nx]}{N}\right) - f(x) \right| \leqslant \frac{Nx-[Nx]}{N} \leqslant \frac{1}{N}.$$

If the function f is, in addition, a monotone increasing (or decreasing) function, we can obtain a smaller error estimate just by adding the number $1/(2N)$(or $-1/(2N)$) to each element of the table. More precisely, we want to show that for a monotone increasing Lipschitz function f on $[0, 1]$ we have

$$\left| f\left(\frac{[Nx]}{N}\right) + \frac{1}{2N} - f(x) \right| \leqslant \frac{1}{2N}. \tag{2}$$

For a monotone decreasing Lipschitz function f on $[0, 1]$ we have

$$\left| f\left(\frac{[Nx]}{N}\right) - \frac{1}{2N} - f(x) \right| \leqslant \frac{1}{2N}. \tag{3}$$

Examples of functions satisfying condition (1) on $[0, 1]$ are $\arctan(x)$, $\sin(x)$, $\cos(x)$, $\exp(-x)$, and many others. To obtain values of increasing functions with an error estimate of 0.005, we have to construct a table of 101 numbers

$$f(0)+0.005,\ f(0.01)+0.005,\ \ldots,\ f(0.99)+0.005,\ f(1)+0.005.$$

The proof of inequality (2) is geometrically obvious. The analytic proof is just as simple. Assuming that f is a monotone increasing function satisfying condition (1) on $[0, 1]$, and using inequalities $x-1 \leqslant [x] \leqslant x$, we have

$$f\left(x-\frac{1}{N}\right) + \frac{1}{2N} \leqslant f\left(\frac{[Nx]}{N}\right) + \frac{1}{2N} \leqslant f(x) + \frac{1}{2N}. \tag{4}$$

By the Lipschitz condition we have

$$f(x) - f\left(x-\frac{1}{N}\right) \leqslant \frac{1}{N}$$

or

$$f(x)-\frac{1}{2N}\leqslant f\left(x-\frac{1}{N}\right)+\frac{1}{2N}. \tag{5}$$

From (4) and (5) we conclude that

$$f(x)-\frac{1}{2N}\leqslant f\left(\frac{[Nx]}{N}\right)+\frac{1}{2N}\leqslant f(x)+\frac{1}{2N}$$

and inequality (2) is proved. The proof of inequality (3) is similar.

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